Supporting grades 4 – 8 students' understanding of standard algorithms while working toward fluency

Sybilla Beckmann

Department of Mathematics
University of Georgia

Kansas MTSS Symposium September 2013

In this session . . .

. . . we will examine some parts of these domains in the Common Core State Standards for Mathematics in grades 4 – 8:

- Operations and Algebraic Thinking
- Numbers and Operations in Base Ten
- The Number System

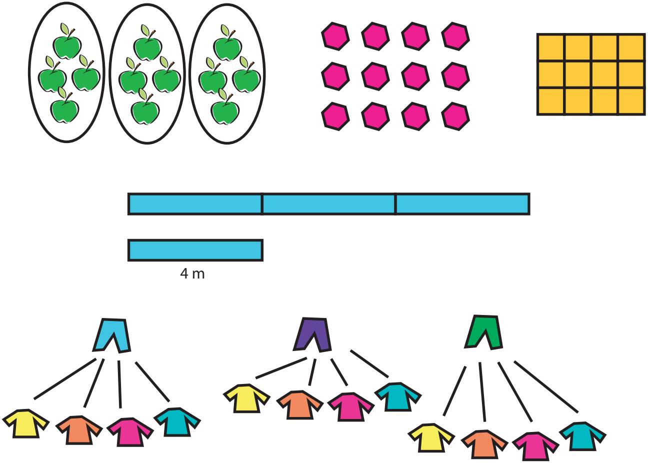
We will focus on ways of fostering understanding while supporting students to move toward fluency.

- Meanings of addition and subtraction (K – 2),
  multiplication and division (3 – 5)
  types of problems these operations solve
    - MP1 Make sense of problems and persevere in solving them
    - MP2 Reason abstractly and quantitatively
    - MP4 Model with mathematics
- Algebraic properties of the operations; other patterns and rules
- Single-digit additions/related subtractions;
  single digit multiplications/related divisions;

  use of properties in *learning* them, *not rotely memorizing them*
    - MP7 Look for an make use of structure
    - MP8 Look for and express regularity in repeated reasoning

Types of multiplication and division problems

3×4



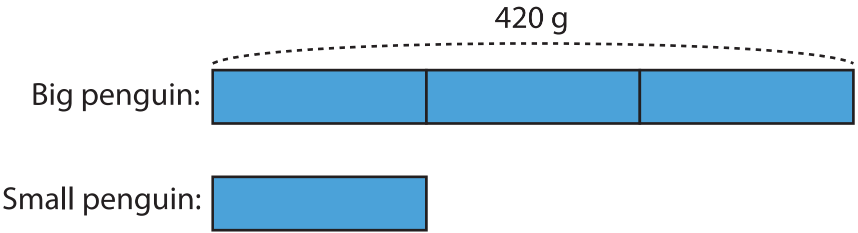
Multiplication and division

$$\begin{array}{ccccc} A & \times & B & = & C \\ \uparrow & & \uparrow & & \uparrow \\ \text{\# of groups} & & \text{amount in one whole group} & & \text{product amount} \end{array}$$

Unknown product	$3 \times 4 = ?$
Group size unknown "How many in each group?" division	$3 \times ? = 12$ $\quad 12 \div 3 = ?$
Number of groups unknown "How many groups?" division	$? \times 4 = 12$ $\quad 12 \div 4 = ?$

Multi-step word problems in Grade 4 and up

A big penguin will eat 3 times as much fish as a small penguin. The big penguin will eat 420 grams of fish. All together, how much will the two penguins eat?



Your turn: write and discuss problems

- Write two types of division problem using the same context (for contrast)
- OR: write three multiplicative comparison problems using "times as much/many" wording
  multiplication and the two types of division;
- OR: write a multi-step problem involving multiplication
- OR: write an "ordered pair" multiplication problem for a compound event (grade 7 probability)

The University of Georgia

Levels in single-digit multiplications and associated divisions

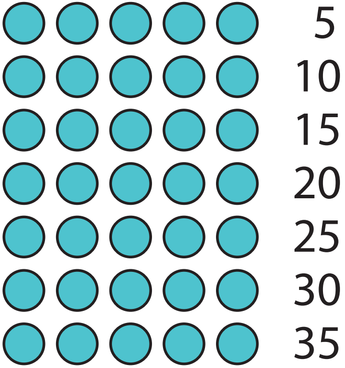
Progression of numerical strategies in working toward fluency — *not just rote memorization of the single-digit facts*:

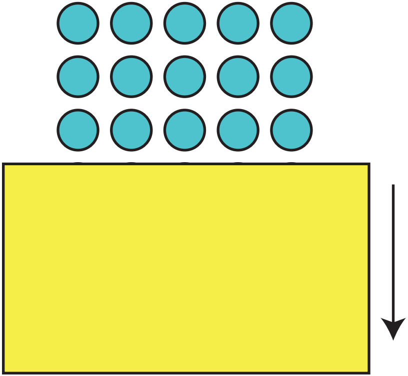
- Level 1: Make and count all (Grade 2)
- Level 2: "Skip counting"
  8×3 count by 3s eight times
  $24 \div 3$ count by 3s until 24 is reached, keeping track of how many counts
- Level 3: Make use of properties of arithmetic: commutative, associative, distributive (perhaps implicitly)
  I know 6×5 is 30, so 7×5 is 5 more, 35 (distributive)

Supported by examining patterns in the multiplication table

The University of Georgia

Level 2: Count by list supported with an array





slide paper down
to uncover rows
while skip counting

Learning single-digit multiplications and associated divisions

Level 3 makes use of properties of multiplication (sometimes implicit).

Is the commutative property of multiplication obvious?

$$A \times B = B \times A$$

for all numbers A, B

The University of Georgia

Learning single-digit multiplications and associated divisions

Level 3 makes use of properties of multiplication (sometimes implicit).

Is the commutative property of multiplication obvious?

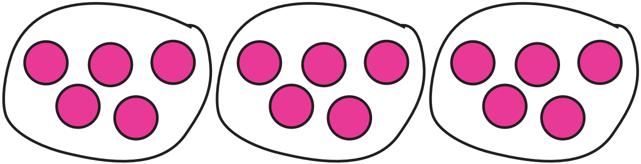
$$A \times B = B \times A$$

for all numbers A, B

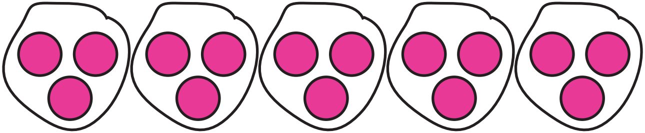
The commutative property of multiplication

A 3rd grade perspective on why the commutative property of multiplication is not obvious:

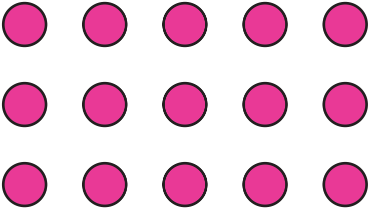
3×5



5×3

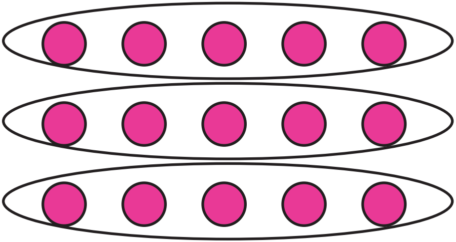


The commutative property of multiplication

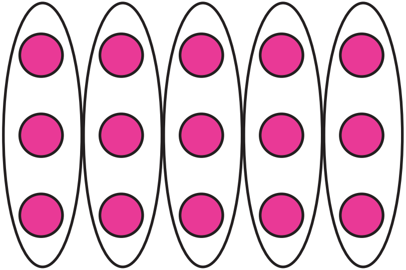


The commutative property of multiplication

3×5



The commutative property of multiplication



5×3

Level 3: Using relationships to derive new facts from other facts

These apply the distributive property (left, right) and the associative property (middle):

$6\times7 = 5\times7 + 1\times7$ $\qquad$ $6\times7 = 2\times(3\times7)$ $\qquad$ $6\times7 = 6\times5 + 6\times2$

Your turn: show how to derive single-digit multiplications from other multiplications

Representing, comparing, and calculating with numbers in base ten.

- Use strategies based on place value and properties of operations
- Illustrate and explain calculations with representations such as mathematical drawings
    - MP2 Reason abstractly and quantitatively
    - MP3 Construct viable arguments and critique the reasoning of others
    - MP5 Use appropriate tools strategically
    - MP6 Attend to precision
    - MP7 Look for and make use of structure
    - MP8 Look for and express regularity in repeated reasoning
- Work towards fluency with understanding

Strategies, the standard algorithms, and written methods

Fuson, K.C. & Beckmann, S. (Fall/Winter 2012-02013). Standard algorithms in the Common Core State Standards. *National Council of of Supervisors of Mathematics Journal of Mathematics Education Leadership, 14*(2), 4 – 30. www.mathedleadership.org

Please see your handout for examples of **written methods** that implement the **standard multiplication algorithm** and the **standard division algorithm**. We will look at those in detail, but first . . .

Strategies, the standard algorithms, and written methods

Computation strategies — thoughtful approaches; the emphasis is on student sense-making.
These include special strategies for particular problems that may not generalize.

For example:

$$98 + 17 = 98 + (2 + 15) = (98 + 2) + 15 = 100 + 15 = 115$$

Or:
To multiply 8×15:
double 15 $\rightarrow$ 30, double 30 $\rightarrow$ 60, double 60 $\rightarrow$ 120

The University of Georgia

Strategies, the standard algorithms, and written methods

Standard algorithms — for each operation there is a particular *mathematical approach* that is based on decomposing numbers into base-ten units and applying properties of operations to reduce a calculation to single-digit calculations together with correct place value placement.

$$\begin{aligned} 8 \times 549 &= 8 \times (500 + 40 + 9) \\ &= (8 \times 500) + (8 \times 40) + (8 \times 9) \\ &= (8 \times 5) \times 100 + (8 \times 4) \times 10 + (8 \times 9) \end{aligned}$$

Strategies, the standard algorithms, and written methods

To implement a standard algorithm we use a systematic **written method** for recording the steps of the algorithm.

There are variations in written methods — some are longer because they include extra steps or math drawings.

Over time the longer written methods can be abbreviated to shorter methods that allow students to achieve fluency with the standard algorithm while still being able to understand and explain the method.

Strategies, the standard algorithms, and written methods

In the past there has been an unfortunate dichotomy suggesting that *strategy* implies understanding and *algorithm* implies no visual models, no explaining, and no understanding.

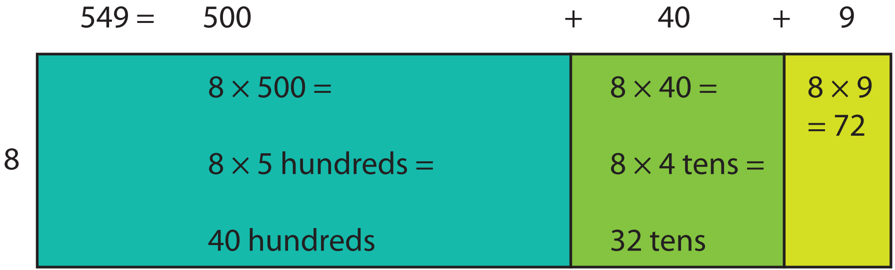
In the past, teaching *the standard algorithm* has too often meant teaching numerical steps rotely and having students memorize steps rather than understand and explain them.

The CCSS clearly do not mean for this to happen!

The University of Georgia

The standard multiplication algorithm

Simplified array/area drawing for 8×549



The standard multiplication algorithm

Three accessible ways to record the standard algorithm:

Left to right showing the partial products

$$\begin{array}{r l} 549 & \\ \times\ 8 & \text{thinking:} \\ \hline 4000 & \boxed{8 \times 5 \text{ hundreds}} \\ 320 & \boxed{8 \times 4 \text{ tens}} \\ 72 & \boxed{8 \times 9} \\ \hline 4392 & \end{array}$$

Right to left showing the partial products

$$\begin{array}{r l} 549 & \\ \times\ 8 & \text{thinking:} \\ \hline 72 & \boxed{8 \times 9} \\ 320 & \boxed{8 \times 4 \text{ tens}} \\ 4000 & \boxed{8 \times 5 \text{ hundreds}} \\ \hline 4392 & \end{array}$$

Right to left recording the carries below

$$\begin{array}{r} 549 \\ \times\ 8 \\ \hline {}^{3\ 7} \\ 4022 \\ \hline 4392 \end{array}$$

Multiplying multiples of 10

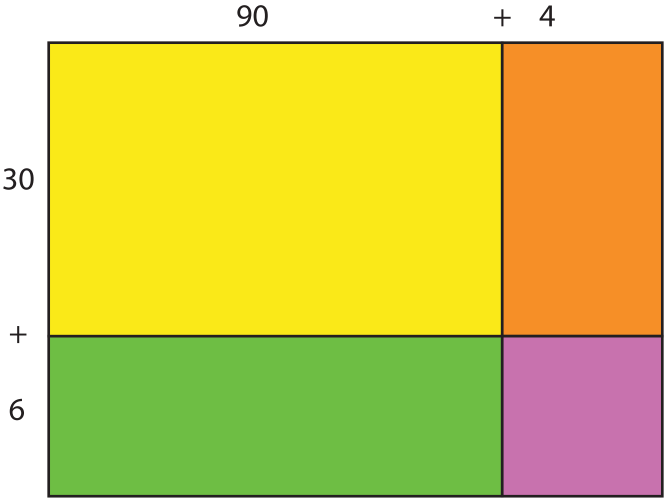
$$3 \times 50$$

3 times (5 tens)
(3 times 5) tens

$$\begin{aligned} 3 \times 50 &= 3 \times (5 \times 10) \\ &= (3 \times 5) \times 10 \\ &= 15 \times 10 = 150 \end{aligned}$$

Uses the associative property of multiplication

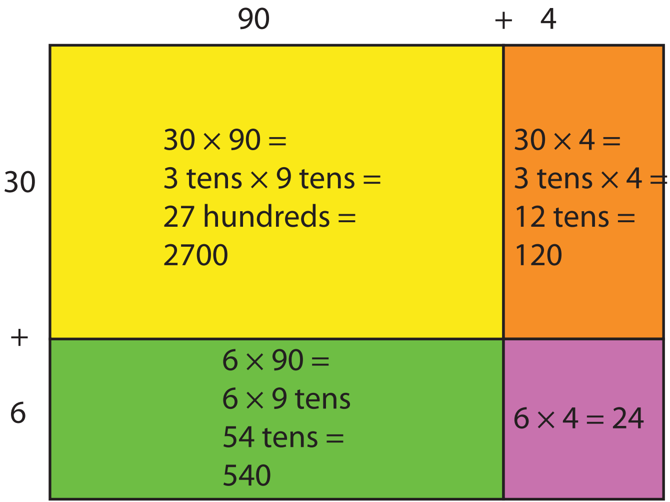
The standard multiplication algorithm



$$\begin{array}{r} 94 \\ \times\, 36 \\ \hline \end{array}$$

The University of Georgia

The standard multiplication algorithm



$$\begin{array}{r} 94 \\ \times\ 36 \\ \hline \boxed{24} \\ \boxed{540} \\ \boxed{120} \\ \boxed{2700} \\ \hline 3384 \end{array}$$

The standard multiplication algorithm

Two accessible, right to left ways to record the standard algorithm:

Showing the partial products

$$
\begin{array}{r}
94 \\
\times\ 36 \\
\hline
24 \\
540 \\
120 \\
2700 \\
\hline
{\scriptstyle 1}\quad\ \ \\
3384
\end{array}
$$

thinking:

$\boxed{6 \times 4}$

$\boxed{6 \times 9 \text{ tens}}$

$\boxed{3 \text{ tens} \times 4}$

$\boxed{3 \text{ tens} \times 9 \text{ tens}}$

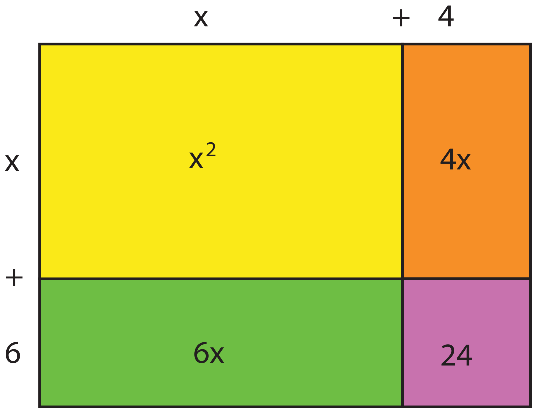
Recording the carries below for $\boxed{\text{correct place value placement}}$

$$
\begin{array}{r}
94 \\
\times\ 36 \\
\hline
{\scriptstyle 5\ \ 2}\ \ \\
44 \\
\boxed{\scriptstyle 2\ \ 1}\ \ \ \\
72\boxed{0} \\
\hline
{\scriptstyle 1}\qquad\ \ \\
3384
\end{array}
$$

$\boxed{0}$ because we are multiplying by 3 tens in this row

We use the same ideas in algebra

$$(x+6)(x+4) = x^2 + 4x + 6x + 24$$



Your turn: discuss written methods

Discuss the different written methods shown on your handout for the **standard multiplication algorithm.**
Try them for some other numbers!

Discuss the written methods shown on your handout for the **standard division algorithm.**
Try them for some other numbers!

The standard division algorithm

Next: other conceptual language for the standard division algorithm.

For $745 \div 3$ think of 745 objects bundled into:
7 bundles of 100, each made of 10 bundles of 10
4 bundles of 10
5 ones

Divide the objects equally among 3 groups, starting with the hundreds bundles.

The standard division algorithm

$$745 \div 3 = ?$$

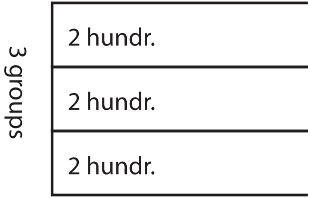
$3\overline{)745}$

Thinking:

Divide
7 hundreds, 4 tens, 5 ones
equally among 3 groups,
starting with hundreds.

The standard division algorithm

$$745 \div 3 = ?$$

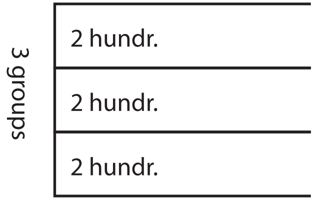


$$\begin{array}{r} 2 \\ 3\,\overline{)\,745} \\ -6 \\ \hline 1 \end{array}$$

Thinking:

7 hundreds ÷ 3
each group gets
2 hundreds;
1 hundred is left.

$$745 \div 3 = ?$$




```
    2
3 )745
  - 6
   14
```

Thinking:

Unbundle 1 hundred.
Now I have
10 tens + 4 tens
= 14 tens.

The standard division algorithm

$$745 \div 3 = ?$$

3 groups

2 hundr. + 4 tens
2 hundr. + 4 tens
2 hundr. + 4 tens

$$\begin{array}{r} 24 \\ 3\,\overline{)745} \\ -6 \\ \hline 14 \\ -12 \\ \hline 2 \end{array}$$

Thinking:

14 tens ÷ 3
each group gets
4 tens;
2 tens are left.

The University of Georgia

The standard division algorithm

$$745 \div 3 = ?$$

3 groups

2 hundr. + 4 tens
2 hundr. + 4 tens
2 hundr. + 4 tens

$$\begin{array}{r} 24 \\ 3\,\overline{)\,745} \\ -\,6 \\ \hline 14 \\ -\,12 \\ \hline 25 \end{array}$$

Thinking:

Unbundle 2 tens.
Now I have
$20 + 5 = 25$ left.

The standard division algorithm

$$745 \div 3 = ?$$

3 groups

2 hundr. + 4 tens + 8
2 hundr. + 4 tens + 8
2 hundr. + 4 tens + 8

Thinking:

25 ÷ 3
each group gets 8;
1 is left.

$$\begin{array}{r} 248 \\ 3\,\overline{)745} \\ -6 \\ \hline 14 \\ -12 \\ \hline 25 \\ -24 \\ \hline 1 \end{array}$$

Each group got 248
and 1 is left.

The University of Georgia

Your turn

Use the idea of dividing bundled objects among groups to explain the standard algorithm for $745 \div 3$.

- Work place by place so that you only work with one type of bundle at a time;
- When you unbundle, you replace each bundle with 10 bundles of the next lower place.

Now try some more examples, including ones that have decimal quotients!

Application of division: Converting fractions to decimals

CCSS 7.NS.2.d
"Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats."

How do we know the decimal form of a fraction terminates or eventually repeats?

Division: what to do with the remainder?

1. Ignore the remainder
2. Add 1 to the quotient
3. The remainder is the answer
4. Mixed number answer
5. Decimal answer
6. Use two units in the answer: A 14 foot long piece of rope is divided into 3 equal pieces. How long is each piece? 4 feet, 8 inches

Remainder versus mixed number

14 brownies are put into packages of 3 each. How many packages? 1 package $\longrightarrow$ 3 brownies ? packages $\longrightarrow$ 14 brownies	14 brownies are put equally into 3 packages. How many brownies in each? 3 packs $\longrightarrow$ 14 brownies 1 pack $\longrightarrow$? brownies
4, R 2 4 what? 2 what?	4, R 2 4 what? 2 what?
$4\frac{2}{3}$ 4 what? 2/3 of what?	$4\frac{2}{3}$ 4 what? 2/3 of what?

The University of Georgia

Remainder versus mixed number

14 brownies are put into packages of 3 each. How many packages?	14 brownies are put equally into 3 packages. How many brownies in each?
1 package $\longrightarrow$ 3 brownies ? packages $\longrightarrow$ 14 brownies	3 packs $\longrightarrow$ 14 brownies 1 pack $\longrightarrow$? brownies
4, R 2 4 what? 2 what? 4 packages, 2 brownies left	4, R 2 4 what? 2 what? 4 brownies, 2 brownies left
$4\frac{2}{3}$ 4 what? 2/3 of what? 4 packages, 2/3 of a package	$4\frac{2}{3}$ 4 what? 2/3 of what? 4 brownies, 2/3 of a brownie

Thank you!

Questions? Comments?

Find me on Twitter at SybillaBeckmann

Join the Mathematics Teaching Community online at https://mathematicsteachingcommunity.math.uga.edu/

Some resources:

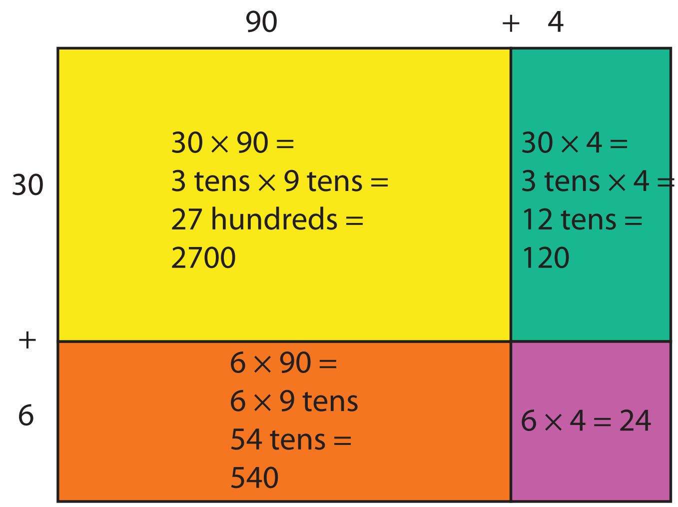
- Fuson, K.C. & Beckmann, S. (Fall/Winter 2012-02013). Standard algorithms in the Common Core State Standards. *National Council of of Supervisors of Mathematics Journal of Mathematics Education Leadership, 14*(2), 4 – 30. www.mathedleadership.org
- The Progressions for the Common Core State Standards: http://ime.math.arizona.edu/progressions/
- *Mathematics for Elementary Teachers* by Sybilla Beckmann

The University of Georgia

Written Methods for the Standard Multiplication Algorithm, 2-digit × 2-digit

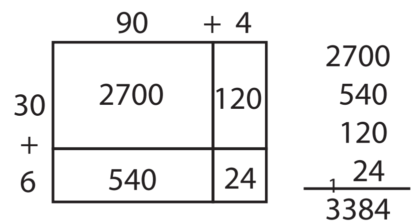
Adapted from: Fuson, K. C. & Beckmann, S. (Fall/Winter 2012-2013). Standard algorithms in the Common Core State Standards. *National Council of Supervisors of Mathematics Journal of Mathematics Education Leadership,14(2),* 4 - 30. www.mathedleadership.org

Array/area drawing for 36 × 94

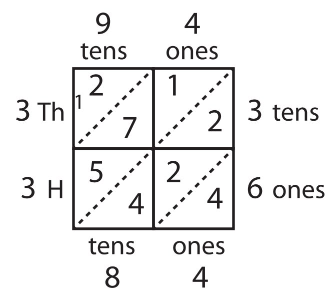


$$36 \times 94 = (30 + 6)\times(90 + 4)$$
$$= 30\times90 + 30\times4 + 6\times90 + 6\times4$$

Area Method 1:



Lattice Method 5:



Method 2:

Showing the partial products

$$\begin{array}{r} 94 \\ \times\, 36 \\ \hline 24 \\ 540 \\ 120 \\ 2700 \\ \hline 3384 \end{array}$$

thinking:

6 × 4

6 × 9 tens

3 tens × 4

3 tens × 9 tens

Method 3:

Recording the carries below for $\boxed{\text{correct place value placement}}$

$$\begin{array}{r} 94 \\ \times\, 36 \\ \hline {}^{5\,2} \\ 44 \\ \boxed{2\ 1} \\ 72\boxed{0} \\ \hline 3384 \end{array}$$

$\boxed{0}$ because we are multiplying by 3 tens in this row

Method 4:

$$\begin{array}{r} 94 \\ \times\, 36 \\ \hline \cancel{2} \\ 564 \\ \cancel{1} \\ 2820 \\ \hline 3384 \end{array}$$

Traditional Method 6:

$$\begin{array}{r} \cancel{1} \\ \cancel{2} \\ 94 \\ \times\, 36 \\ \hline 564 \\ 1 \\ 2820 \\ \hline 3384 \end{array}$$

Written Methods 2 and 3 are shown from right to left, but could go from left to right.

In Methods 3 and 4, digits that represent newly composed tens and hundreds in the partial products are written below the line intead of above 94. This way, the 1 from 30 × 4 = 120 is placed correctly in the hundreds place, unlike in Traditional Method 6, where it is placed in the (incorrect) tens place. In Method 4, the 2 tens from 6 × 4 = 24 are added to the 4 tens from 6 × 90 = 540 and then crossed out so they will not be added again; the situation is similar for the 1 hundred from 30 × 4 = 120.

In Method 3, all multiplying is done first and then all adding. In Method 4 and Traditional Method 6, multiplying and adding alternate, which is more difficult for some students.

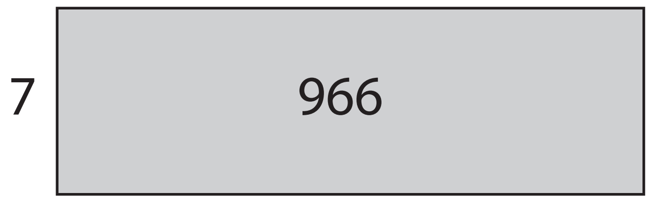
Note that the 0 in the ones place of the second line of Methods 3, 4, and 6 is there because the whole line of digits is produced by multiplying by 30 (not 3).

Written methods for the standard division algorithm, 1-digit divisor

Adapted from: Fuson, K. C. & Beckmann, S. (Fall/Winter 2012-2013). Standard algorithms in the Common Core State Standards. *National Council of Supervisors of Mathematics Journal of Mathematics Education Leadership,14(2),* 4 - 30. www.mathedleadership.org

Area/array drawing for 966 ÷ 7

? hundreds + ? tens + ? ones

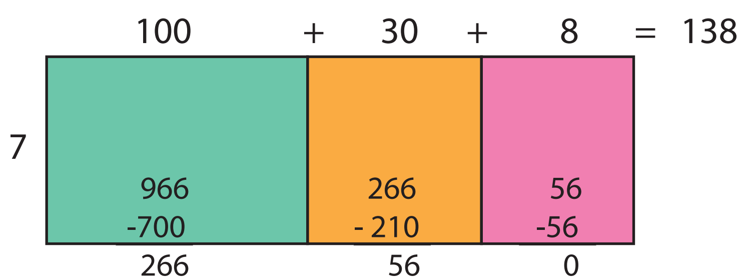


$$7\overline{)966}^{\,???}$$

Thinking: A rectangle has area 966 and one side of length 7. Find the unknown side length. Find hundreds first, then tens, then ones.

$$\begin{aligned}966 &= 7\times100 + 7\times30 + 7\times8\\ &= 7\times(100 + 30 + 8)\\ &= 7\times138\end{aligned}$$

Method A:



8
30
100 } 138
7)966
- 700
266
- 210
56
- 56
0

Method B:

138
7)966
- 7
26
- 21
56
- 56
0

Conceptual language for this method:

Find the unknown length of the rectangle;
first find the hundreds, then the tens, then the ones.

The length gets 1 hundred (units); 2 hundreds (square units) remain.

2 hundreds + 6 tens = 26 tens.

The length gets 3 tens (units); 5 tens (square units) remain.

5 tens + 6 ones = 56 ones.

The length gets 8 ones; 0 remains.

The "bringing down" steps represent unbundling a remaining amount and combining it with the amount at the next lower place.